HIGHER ORDER RIESZ TRANSFORMS FOR LAGUERRE EXPANSIONS

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Abstract. In this paper we investigate L^p-boundedness properties for the higher order Riesz transforms associated with Laguerre operators. Also we prove that the k-th Riesz transform is a principal value singular integral operator (modulus a constant times of the function when k is even). To establish our results we exploit a new identity connecting Riesz transforms in the Hermite and Laguerre settings.

1. Introduction

The aim of this paper is to investigate higher order Riesz transforms associated with Laguerre function expansions. To achieve our goal we use a procedure that will be described below and that was developed for the first time by the authors and Torrea in [4]. Our results complete and improve in some senses the ones obtained by Graczyk, Loeb, López, Nowak and Urbina [11] about higher order Riesz transforms for Laguerre expansions.

For every $\alpha > -1$ we consider the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right), \quad x \in (0,\infty).$$

This operator can be factorized as follows

$$L_\alpha = \frac{1}{2}\mathfrak{D}_\alpha^*\mathfrak{D}_\alpha + \alpha + 1, \tag{1.1}$$

where $\mathfrak{D}_\alpha f = \left(-\dfrac{\alpha+1/2}{x} + x + \dfrac{d}{dx}\right) f = x^{\alpha+\frac{1}{2}}\dfrac{d}{dx}(x^{-\alpha-\frac{1}{2}}f) + xf$, and $\mathfrak{D}_\alpha^*$ denotes the formal adjoint of $\mathfrak{D}_\alpha$ in $L^2((0,\infty),dx)$.

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The heat semigroup $\{W_t^\alpha\}_{t>0}$ generated by the Laguerre operator $-L_\alpha$ admits the integral representation

$$W_t^\alpha(f)(x) = \int_0^\infty W_t^\alpha(x,y)f(y)dy, \quad x \in (0,\infty),\ f \in L^2((0,\infty),dx)\ , \tag{1.2}$$

where the heat kernel $W_t^\alpha(x,y)$, $t,x,y \in (0,\infty)$, is defined in (1.4). The factorization (1.1) for L_α suggests to define (formally), for every $k \in \mathbb{N}$, the k-th Riesz transform $R_\alpha^{(k)}$ associated with L_α by

$$R_\alpha^{(k)} = \mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}}\ .$$

Here $L_\alpha^{-\beta}$, $\beta > 0$, denotes the $-\beta$ power of the operator L_α (see (1.5)).

In the main result of this paper (Theorem 1.3) we prove that the space $C_c^\infty(0,\infty)$ of C^∞ functions on $(0,\infty)$ having compact support on $(0,\infty)$ is contained in the domain of $R_\alpha^{(k)}$ and that $R_\alpha^{(k)}$ on $C_c^\infty(0,\infty)$ is a principal value integral operator. Moreover we establish that $R_\alpha^{(k)}$ can be extended as a principal value integral and bounded operator on certain weighted L^p -spaces.

Theorem 1.3. *Let $\alpha > -1$ and $k \in \mathbb{N}$. For every $\phi \in C_c^\infty(0,\infty)$ it has that*

$$R_\alpha^{(k)}\phi(x) = w_k\phi(x) + \lim_{\varepsilon\to 0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)\phi(y)dy, \quad x \in (0,\infty),$$

where

$$R_\alpha^{(k)}(x,y) = \frac{1}{\Gamma\left(\frac{k}{2}\right)}\int_0^\infty t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt, \quad x,y \in (0,\infty),$$

and $w_k = 0$, when k is odd and $w_k = -2^{\frac{k}{2}}$, when k is even.

The operator $R_\alpha^{(k)}$ can be extended, defining it by

$$R_\alpha^{(k)}f(x) = w_kf(x) + \lim_{\varepsilon\to 0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)f(y)dy, \qquad a.e.\ x \in (0,\infty),$$

as a bounded operator from $L^p((0,\infty),x^\delta dx)$ into itself, for $1 < p < \infty$ and

(a) $-\left(\alpha+\frac{3}{2}\right)p - 1 < \delta < \left(\alpha+\frac{3}{2}\right)p - 1$, when k is odd;

(b) $-\left(\alpha+\frac{1}{2}\right)p - 1 < \delta < \left(\alpha+\frac{3}{2}\right)p - 1$, when k is even;

and as a bounded operator from $L^1((0,\infty),x^\delta dx)$ into $L^{1,\infty}((0,\infty),x^\delta dx)$ when

(c) $-\alpha - \frac{5}{2} \le \delta \le \alpha + \frac{1}{2}$, when k is odd;

(d) $-\alpha - \frac{3}{2} \le \delta \le \alpha + \frac{1}{2}$, for $\alpha \neq -\frac{1}{2}$, and $-1 < \delta \le 1$, for $\alpha = -\frac{1}{2}$, when k is even.

Also we get the corresponding property in the Hermite context (see Proposition 2.2) which completes in the one dimensional case the results in [30] about the higher order Riesz transform associated with the Hermite operator.

For every $n \in \mathbb{N}$, we have that $L_\alpha \varphi_n^\alpha = (2n+\alpha+1)\varphi_n^\alpha$, where,

$$\varphi_n^\alpha(x) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2}} x^{\alpha+\frac{1}{2}} L_n^\alpha(x^2), \quad x \in (0,\infty),$$

and L_n^α denotes the n-th Laguerre polynomial of type α ([31, p. 100] and [32, p. 7]). For every $n \in \mathbb{N}$, φ_n^α is usually called the n-th Laguerre function of type α. The system $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2((0,\infty),dx)$.

The heat semigroup $\{W_t^\alpha\}_{t>0}$ generated by the operator $-L_\alpha$ is defined by

$$W_t^\alpha(f) = \sum_{n=0}^{\infty} e^{-t(2n+\alpha+1)} c_n^\alpha(f)\varphi_n^\alpha, \quad f \in L^2((0,\infty),dx),$$

where $c_n^\alpha(f) = \displaystyle\int_0^\infty \varphi_n^\alpha(x) f(x)dx$, $n \in \mathbb{N}$.

For every $t > 0$ the operator W_t^α admits the integral representation (1.2) where, by the Mehler formula [32, p. 8], for every $t, x, y \in (0,\infty)$,

$$\begin{aligned} W_t^\alpha(x,y) &= \sum_{n=0}^{\infty} e^{-t(2n+\alpha+1)} \varphi_n^\alpha(x)\varphi_n^\alpha(y) \\ &= \left(\frac{2e^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} \left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}}. \end{aligned} \tag{1.4}$$

Here I_α represents the modified Bessel function of the first kind and order α.

According to the ideas presented by Stein [27] the fundamental operators of the harmonic analysis (fractional integrals, Riesz transforms, g-functions,...) can be considered in the context of the Laguerre operator L_α. It is convenient to mention that this way to describe harmonic operators in the Laguerre context was initiated by Muckenhoupt ([19] and [21]).

If $\beta > 0$ the negative power $L_\alpha^{-\beta}$ of L_α is defined by

$$L_\alpha^{-\beta} f = \sum_{n=0}^{\infty} \frac{c_n^\alpha(f)}{(2n+\alpha+1)^\beta}\varphi_n^\alpha, \quad f \in L^2((0,\infty),dx).$$

It is not hard to see that $L_\alpha^{-\beta}$ can be expressed, for every $f \in L^2((0,\infty),dx)$, by means of the following integral

$$L_\alpha^{-\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} W_t^\alpha(f)(x)dt, \qquad \text{a.e. } x \in (0,\infty). \tag{1.5}$$

$L_\alpha^{-\beta}$ is also called β-th fractional integral associated with L_α. This kind of fractional integrals has been investigated by several authors ([6], [10], [11], [17] and [28]).

First order Riesz transforms in the L_α-setting were studied in [24] for $\alpha \geq -\frac{1}{2}$ and in [1] for $\alpha > -1$. Also, the procedure developed in [15] can be used to investigate strong, weak and restricted weak type with respect to the measure $x^\delta dx$ on $(0,\infty)$ for the Riesz transforms $R_\alpha^{(1)}$.

As it was mentioned, in this paper we establish boundedness properties for $R_\alpha^{(k)}$ in $L^p((0,\infty), x^\delta dx)$. In the Laguerre-polynomial context, Graczyk, Loeb, López, Nowak and Urbina [11] investigated the corresponding higher order Riesz transform $\mathcal{R}_\alpha^{(k)}$, $k \in \mathbb{N}$. They used methods that impose substancial restrictions to the admisible values of the index α. Their proofs are based in a connection between n-dimensional Hermite functions and Laguerre functions of order $\alpha = \frac{n}{2} - 1$. This fact forces to consider only half-integer values for α (see [11, Section 4]). This connection between n-dimensional Hermite and Laguerre functions was exploited earlier by Gutiérrez, Incognito and Torrea [12] and Harboure, Torrea and Viviani [16], amongst others. As it is emphatized in [11], the extension of the results of L^p-boundedness for $\mathcal{R}_\alpha^{(k)}$ to all values of $\alpha > -1$ is not an easy problem and it requires a more subtle approach than the one followed in [11].

Our procedure here is completely different from the one used in [11]. In a first step we split the operators $R_\alpha^{(k)}$ into two parts, namely: a local operator and a global one. These operators are integral operators defined by kernels supported close to and far from the diagonal, respectively. The global operator is upper bounded by Hardy type operators. The novelty of our method is the way followed to study the local part. We establish a pointwise identity connecting the kernel of $R_\alpha^{(k)}$ with the one corresponding to the k-th Riesz transform associated with the Hermite operator in one dimension, for every $\alpha > -1$ (see Proposition 3.11). By using this identity we transfer boundedness and convergence results from k-th Riesz transform for Hermite operator in one dimension to k-th Riesz transform in the L_α-setting.

In the literature (see, for instance [5] and [28]) we can find other systems of Laguerre functions different from $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. In particular, from the Laguerre polynomials $\{L_n^\alpha\}_{n\in\mathbb{N}}$ we can derive also the systems $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ and $\{l_n^\alpha\}_{n\in\mathbb{N}}$, where, for every $n \in \mathbb{N}$,

$$\mathcal{L}_n^\alpha(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^\alpha(x), \quad x \in (0,\infty),$$

and

$$l_n^\alpha(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x}{2}} L_n^\alpha(x), \quad x \in (0,\infty)\ .$$

$\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\infty),dx)$ and $\{l_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\infty),x^\alpha dx)$.

As it is shown in [1], harmonic analysis operators associated with $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ and $\{l_n^\alpha\}_{n\in\mathbb{N}}$ is closely connected with the corresponding operators related to the family $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. The connection is given by a multiplication operator defined by $M_\beta f = x^\beta f$, for certain $\beta \in \mathbb{R}$. From the strong type results for $R_\alpha^{(k)}$ established in Theorem 1.3, the corresponding results for the k-th Riesz transform in the $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ and $\{l_n^\alpha\}_{n\in\mathbb{N}}$ settings can be deduced. Moreover the weak type results for the k-th Riesz transform associated with $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ and $\{l_n^\alpha\}_{n\in\mathbb{N}}$ can be obtained by proceeding as in the $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ case of Theorem 1.3.

The organization of the paper is the following. Section 2 contains some basic facts needed in the sequel. Section 3 is devoted to prove the main result of this paper (Theorem 1.3) where we establish that the higher order Riesz transforms are principal value singular integral operators (modulus a constant times of the function, when k is even) and we show $L^p((0,\infty),x^\delta dx)$-boundedness properties for them.

Throughout this paper $C_c^\infty(I)$ denotes the space of functions in $C^\infty(I)$ having compact support on I. By C and c we always represent positive constants that can change from one line to the other one, and $E[r]$, $r \in \mathbb{R}$, stands for the integer part of r.

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2. Preliminaries

In this section we recall some definitions and properties that will be useful in the sequel. By H we denote the Hermite differential operator

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) = -\frac{1}{4}\left[\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right) + \left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)\right]\ . \tag{2.1}$$

Note that $\frac{d}{dx} + x$ and $-\frac{d}{dx} + x$ are formal adjoint operators in $L^2(\mathbb{R},dx)$. Moreover, if $n \in \mathbb{N}$, H_n represents the n-th Hermite polynomial ([31, p. 104]) and h_n is the Hermite function given

by $h_n(x) = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x)$, $x \in \mathbb{R}$, then it has that

$$Hh_n = \left(n + \frac{1}{2}\right) h_n, \quad n \in \mathbb{N} .$$

Moreover, the system $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R}, dx)$.

The investigations of harmonic analysis in the Hermite setting were begun by Muckenhoupt ([19]). This author considered Hermite polynomial expansions instead of Hermite function expansions. In the last decades several authors have studied harmonic analysis operators in the Hermite (polynomial or function) context (see, for instance, [7], [8], [9], [13], [14], [25], [26], [29], [30] and [33]).

The heat semigroup $\{W_t\}_{t>0}$ generated by the operator $-H$ is defined by

$$W_t(f) = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})t} c_n(f) h_n, \quad f \in L^2(\mathbb{R}, dx) \text{ and } t > 0,$$

where $c_n(f) = \displaystyle\int_{-\infty}^{+\infty} h_n(x) f(x) dx$, $n \in \mathbb{N}$ and $f \in L^2(\mathbb{R}, dx)$.

For every $t > 0$, the operator W_t can be described by the integral

$$W_t(f)(x) = \int_{-\infty}^{+\infty} W_t(x,y) f(y) dy, \quad x \in \mathbb{R} \text{ and } f \in L^2(\mathbb{R}, dx),$$

where, according to the Mehler formula [32, p. 2], we have that, for each $x, y \in \mathbb{R}$ and $t > 0$,

$$W_t(x,y) = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})t} h_n(x) h_n(y) = \frac{1}{\sqrt{\pi}} \left(\frac{e^{-t}}{1 - e^{-2t}}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}} + \frac{2xye^{-t}}{1-e^{-2t}}}.$$

The negative power $H^{-\beta}$, $\beta > 0$, of H is given by

$$H^{-\beta} f = \sum_{n=0}^{\infty} \frac{c_n(f)}{\left(n + \frac{1}{2}\right)^{\beta}} h_n, \quad f \in L^2(\mathbb{R}, dx).$$

It can be seen that, for every $f \in L^2(\mathbb{R}, dx)$,

$$H^{-\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} t^{\beta-1} W_t(f)(x) dt, \quad \text{a.e. } x \in \mathbb{R} ,$$

and that, when $\phi \in C_c^{\infty}(\mathbb{R})$,

$$H^{-\beta}\phi(x) = \int_{-\infty}^{+\infty} K_{2\beta}(x,y)\phi(y) dy, \quad x \in \mathbb{R} .$$

Here, for every $\gamma > 0$,

$$K_\gamma(x,y) = \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^{\infty} t^{\frac{\gamma}{2}-1} W_t(x,y) dt, \quad x, y \in \mathbb{R} .$$

The factorization in (2.1) suggests to define the Riesz transform R associated with H by

$$Rf = \left(\frac{d}{dx} + x\right) H^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \left(\frac{2n}{n+\frac{1}{2}}\right)^{\frac{1}{2}} c_n(f) h_{n-1}, \quad f \in L^2(\mathbb{R}, dx) \ .$$

The operator R admits the integral representation

$$Rf(x) = \int_{-\infty}^{+\infty} R(x,y) f(y) dy, \quad x \in \mathbb{R} \setminus \operatorname{supp} f \text{ and } f \in L^2(\mathbb{R}, dx),$$

where

$$R(x,y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{1}{2}} \Big(\frac{d}{dx} + x\Big) W_t(x,y) dt, \quad x, y \in \mathbb{R} \ .$$

L^p-boundedness properties of the Riesz transform R (even in the n-dimensional case) were established in [29].

In [30] higher order Riesz transforms in the Hermite function setting on $\mathbb{R}^n$ were investigated. For every $k \in \mathbb{N}$ the k-th Riesz transforms on $\mathbb{R}$, $R^{(k)}$, is defined by

$$R^{(k)} f = \Big(\frac{d}{dx} + x\Big)^k H^{-\frac{k}{2}} f = \sum_{n=k}^{\infty} \frac{2^{\frac{k}{2}} (n(n-1)...(n-k+1))^{\frac{1}{2}}}{(n+\frac{1}{2})^{\frac{k}{2}}} c_n(f) h_{n-k}, \quad f \in C_c^{\infty}(\mathbb{R}).$$

L^p-boundedness properties of the Riesz transform $R^{(k)}$ were established in [30, Theorem 2.3] by invoking Calderón-Zygmund singular integral theory. For every $k \in \mathbb{N}$, $R^{(k)}$ can be extended to $L^p(\mathbb{R}, dx)$ as a bounded operator from $L^p(\mathbb{R}, dx)$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}, dx)$ into $L^{1,\infty}(\mathbb{R}, dx)$. It is remarkable to note that the L^p-mapping properties for the higher order Riesz transform in the Hermite polynomial setting are essentially different to the corresponding ones in the Hermite function context ([8] and [9]).

We shall not use the Calderón-Zygmund singular integral theory to investigate the higher order Riesz transforms associated with the Laguerre operator. As it was mentioned we shall exploit a connection between higher order Riesz transforms in the Hermite and Laguerre settings.

The following new property that will be established in Section 3 is needed in the proof of Theorem 1.3. It states that the higher order Riesz transform for the Hermite operator is actually a principal value integral operator.

Proposition 2.2. *Let $k \in \mathbb{N}$. Then, for every $f \in C_c^{\infty}(\mathbb{R})$, $1 \leq p < \infty$,*

$$R^{(k)} f(x) = w_k f(x) + \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R^{(k)}(x,y) f(y) dy, \quad a.e. \ x \in \mathbb{R}, \tag{2.3}$$

where

$$R^{(k)}(x,y) = \frac{1}{\Gamma(\frac{k}{2})}\int_0^\infty t^{\frac{k}{2}-1}\left(\frac{d}{dx}+x\right)^k W_t(x,y)dt, \quad x,y\in\mathbb{R},$$

and $w_k = 0$, *when* k *is odd, and* $w_k = -2^{\frac{k}{2}}$, *when* k *is even.*

Since $R^{(k)}(x,y)$, $x,y\in\mathbb{R}$, is a Calderón-Zygmund kernel ([30]) by using standard density arguments from Proposition 2.2 we deduce that the operator $R^{(k)}$ can be extended by (2.3) to $L^p(\mathbb{R},dx)$, $1\le p<\infty$, as a bounded operator from $L^p(\mathbb{R},dx)$ into itself, $1<p<\infty$, and from $L^1(\mathbb{R},dx)$ into $L^{1,\infty}(\mathbb{R},dx)$.

As it was indicated the modified Bessel function I_α of the first kind and order α appears in the kernel of the heat semigroup generated by the Laguerre operator $-L_\alpha$. The following properties of the function I_α will be repeteadly used in the sequel (see [18] and [34]):

(P1) $I_\alpha(z)\sim z^\alpha$, $z\to 0$.

(P2) $\sqrt{z}I_\alpha(z) = \dfrac{e^z}{\sqrt{2\pi}}\left(\displaystyle\sum_{r=0}^{n}(-1)^r[\alpha,r](2z)^{-r} + O(z^{-n-1})\right)$, $n=0,1,2,...$, where $[\alpha,0]=1$ and

$$[\alpha,r] = \frac{(4\alpha^2-1)(4\alpha^2-3^2)\cdots(4\alpha^2-(2r-1)^2)}{2^{2r}\Gamma(r+1)}, \quad r=1,2,\dots .$$

(P3) $\dfrac{d}{dz}(z^{-\alpha}I_\alpha(z)) = z^{-\alpha}I_{\alpha+1}(z)$, $z\in(0,\infty)$.

On the other hand, in our study for the global part of the operators we consider the Hardy type operators defined by

$$H_0^\eta(f)(x) = x^{-\eta-1}\int_0^x y^\eta f(y)dy, \quad x\in(0,\infty),$$

and

$$H_\infty^\eta(f)(x) = x^\eta\int_x^\infty y^{-\eta-1}f(y)dy, \quad x\in(0,\infty),$$

where $\eta>-1$. L^p-boundedness properties of the operators H_0^η and H_∞^η were established by Muckenhoupt [22] and Andersen and Muckenhoupt [2]. In particular, mappings properties for H_0^η and H_∞^η on $L^p((0,\infty),x^\delta dx)$ can be encountered in [5, Lemmas 3.1 and 3.2].

The following formula established in [11, Lemma 4.3, (4.6)] will be frequently used in the sequel. For every $N\in\mathbb{N}$, and a sufficiently smooth function $g:(0,\infty)\longrightarrow\mathbb{R}$, it has that

$$\frac{d^N}{dx^N}[g(x^2)] = \sum_{l=0}^{E[\frac{N}{2}]} E_{N,l}x^{N-2l}\left(\frac{d^{N-l}}{dx^{N-l}}g\right)(x^2) \tag{2.4}$$

where

$$E_{N,l} = 2^{N-2l} \frac{N!}{l!(N-2l)!}, \quad 0 \le l \le E\Big[\frac{N}{2}\Big].$$

We finish this section establishing the following technical lemma that is needed in the proof of Proposition 3.11.

Lemma 2.5. *Let $\alpha > -1$ and $j \in \mathbb{N}$, $j \geq 1$. For every $m = 0, 1, \dots, E[\frac{j}{2}]$, we have*

$$\sum_{n=0}^{m} \sum_{l=2n}^{j} (-1)^{l+n} \binom{j}{l} \frac{E_{l,n}}{2^{l-2n}} [\alpha + l - n, m - n] = 0. \tag{2.6}$$

Proof. For every $s = 0, ..., j$ we denote by $A_{j,s}$ the values

$$A_{j,s} = \sum_{l=0}^{j} (-1)^l \binom{j}{l} l^s,$$

where we take the convention $0^0 = 1$.

In [3, (43)] it was established that, for every $j \in \mathbb{N}$, $j \geq 1$,

$$A_{j,s} = 0, \quad s = 0, 1, \cdots, j-1. \tag{2.7}$$

On the other hand, since $\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}$, for $m \geq n \geq 1$, by using (2.7) we obtain that $A_{j,j} = -jA_{j-1,j-1}$, $j \in \mathbb{N}$, $j \geq 1$ and so $A_{j,j} = (-1)^j j!$, $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$, $j \geq 1$ and consider $m = 0, 1, ..., E[\frac{j}{2}]$ and $n = 0, 1, ..., m$. We can write

$$\begin{aligned}
\sum_{l=2n}^{j} (-1)^l \binom{j}{l} \frac{E_{l,n}}{2^{l-2n}} [\alpha + l - n, m - n] &= \sum_{s=0}^{j-2n} (-1)^s \binom{j}{s+2n} \frac{E_{s+2n,n}}{2^s} [\alpha + s + n, m - n] \\
&= \frac{j!}{n!} \sum_{s=0}^{j-2n} \frac{(-1)^s}{s!(j-2n-s)!} [\alpha + s + n, m - n] \\
&= \frac{j!}{n!(j-2n)!} \sum_{s=0}^{j-2n} (-1)^s \binom{j-2n}{s} [\alpha + s + n, m - n]\ .
\end{aligned}$$

We observe that $[\alpha + s + n, m - n]$ is a polynomial in s which has degree $2(m-n)$. Besides, if j is odd, $2(m-n) \leq j - 2n - 1$. Hence (2.7) allows us to conclude (2.6) in this case.

Assume now that j is even. Then $2(m-n) \leq j - 2n$ and (2.7) leads to

$$\sum_{n=0}^{m} \sum_{l=2n}^{j} (-1)^{l+n} \binom{j}{l} \frac{E_{l,n}}{2^{l-2n}} [\alpha + l - n, m - n] = 0,$$

when $m = 0, 1, ..., \frac{j}{2} - 1$.

For $m = \frac{j}{2}$ and again by (2.7) we can write

$$\sum_{n=0}^{m} \sum_{l=2n}^{j} (-1)^{l+n} \binom{j}{l} \frac{E_{l,n}}{2^{l-2n}} [\alpha + l - n, m - n] = \sum_{n=0}^{\frac{j}{2}} \frac{(-1)^n j!}{n!(j-2n)!} \sum_{s=0}^{j-2n} (-1)^s \binom{j-2n}{s} \frac{s^{j-2n}}{(\frac{j}{2}-n)!}$$

$$= j! \sum_{n=0}^{\frac{j}{2}} \frac{(-1)^n}{n!(j-2n)!(\frac{j}{2}-n)!} A_{j-2n,j-2n} = \frac{j!}{(\frac{j}{2})!} \sum_{n=0}^{\frac{j}{2}} (-1)^n \binom{\frac{j}{2}}{n} = 0\ .$$

Thus (2.6) is established. □

3. Higher order Riesz transforms associated with Laguerre expansions

In this section we prove our main result (Theorem 1.3) concerning to higher order Riesz transforms associated with the sequence $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ of Laguerre functions. As it was mentioned, our procedure is based on certain connection between higher order Riesz transforms in the Laguerre and Hermite settings.

We start proving the following results about the differentiability of $H^{-\frac{k}{2}}$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ and $l = 0, 1, ..., k$, let us consider

$$R^{(k,l)}(x,y) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty t^{\frac{k}{2}-1} \Big(\frac{d}{dx} + x\Big)^l W_t(x,y)dt, \quad x, y \in \mathbb{R}.$$

(Note that $R^{(k,k)}(x,y) = R^{(k)}(x,y)$, $x, y \in \mathbb{R}$).

Proposition 3.1. *Let $\phi \in C_c^\infty(\mathbb{R})$ and $k \in \mathbb{N}$. We have that*

$$\Big(\frac{d}{dx} + x\Big)^l H^{-\frac{k}{2}} \phi(x) = \int_{-\infty}^{+\infty} R^{(k,l)}(x,y)\phi(y)dy,$$

for every $x \in \mathbb{R}$, when $l = 0, ..., k-1$ and for every $x \in \mathbb{R} \setminus \operatorname{supp} \phi$, when $l = k$.

Proof. As it was mentioned, we can write

$$H^{-\frac{k}{2}}\phi(x) = \int_{-\infty}^{+\infty} K_k(x,y)\phi(y)dy, \quad x \in \mathbb{R}.$$

So, in order to establish our result we must analyze the integral

$$\int_0^\infty t^{\frac{k}{2}-1} \left| \Big(\frac{d}{dx} + x\Big)^l (W_t(x,y)) \right| dt, \quad x, y \in \mathbb{R}.$$

A calculation shows that we can write, for every $l \in \mathbb{N}$,

$$\Big(\frac{d}{dx} + x\Big)^l = \sum_{0 \le \rho+\sigma \le l} c_{\rho,\sigma}^l x^\rho \frac{d^\sigma}{dx^\sigma}\ , \tag{3.2}$$

where $c^l_{\rho,\sigma} \in \mathbb{R}$, $\rho, \sigma \in \mathbb{N}$, $0 \le \rho+\sigma \le l$ and $c^l_{0,l} = 1$.

By making the change of variable $t = \log \frac{1+s}{1-s}$, we obtain, for every $s \in (0,1)$ and $x, y \in \mathbb{R}$,

$$\left(\frac{d}{dx}+x\right)^l W_t(x,y) = \frac{1}{\sqrt{\pi}}\left(\frac{1-s^2}{4s}\right)^{\frac{1}{2}} \sum_{0\le\rho+\sigma\le l} c^l_{\rho,\sigma} x^\rho \frac{d^\sigma}{dx^\sigma}\left(e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\right), \quad l \in \mathbb{N}.$$

Moreover, according to [30, p. 50], $\frac{d^\sigma}{dx^\sigma}\left(e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\right)$ is a linear combination of terms of the form

$$\left(s+\frac{1}{s}\right)^{b_1} e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\left(s(x+y)+\frac{1}{s}(x-y)\right)^{b_2},$$

where $b_1, b_2 \in \mathbb{N}$ and $2b_1 + b_2 \le \sigma$.

Assume that $l, \rho, \sigma, b_1, b_2 \in \mathbb{N}$, $0 \le l \le k$, $0 \le \rho+\sigma \le l$ and $2b_1+b_2 \le \sigma$. Let us consider

$$\begin{aligned} I^{b_1,b_2}_{l,\rho,\sigma}(x,y) &= x^\rho \int_0^1 \left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\left(\frac{1-s^2}{s}\right)^{\frac{1}{2}}\left(s+\frac{1}{s}\right)^{b_1} e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)} \\ &\quad \times\left(s(x+y)+\frac{1}{s}(x-y)\right)^{b_2}\frac{1}{1-s^2}ds, \quad x,y\in\mathbb{R}. \end{aligned}$$

We have that

$$\begin{aligned} |I^{b_1,b_2}_{l,\rho,\sigma}(x,y)| &\le C\left(\int_0^{\frac{1}{2}} s^{\frac{k}{2}-\frac{3}{2}-b_1-\frac{b_2}{2}-\frac{\rho}{2}} e^{-c\frac{(x-y)^2}{s}} ds + \int_{\frac{1}{2}}^1 (-\log(1-s))^{\frac{k}{2}-1}\frac{1}{\sqrt{1-s}}ds\right) \\ &\le C\left(\int_0^{\frac{1}{2}} s^{\frac{1}{2}(k-l-3)} e^{-c\frac{(x-y)^2}{s}} ds + 1\right). \end{aligned}$$

Hence, according to [30, Lemma 1.1],

$$|I^{b_1,b_2}_{l,\rho,\sigma}(x,y)| \le C\begin{cases} 1, & 0\le l\le k-2, \\ \dfrac{1}{\sqrt{|x-y|}}, & l = k-1, \\ \dfrac{1}{|x-y|}, & l = k, \end{cases}, \quad x,y\in\mathbb{R},$$

which allows us to conclude the result. $\square$

We now complete the above result proving that, for every $\phi \in C^\infty_c(\mathbb{R})$ and $k \in \mathbb{N}$, $H^{-\frac{k}{2}}\phi$ is k-times differentiable on $\mathbb{R}$ and that $\left(\frac{d}{dx}+x\right)^k H^{-\frac{k}{2}}\phi(x)$ is given by a principal value integral, for every $x \in \mathbb{R}$.

Proposition 3.3. *Let $\phi \in C_c^\infty(\mathbb{R})$ and $k \in \mathbb{N}$. Then*

$$\left(\frac{d}{dx}+x\right)^k H^{-\frac{k}{2}}\phi(x) = w_k\phi(x) + \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} R^{(k)}(x,y)\phi(y)\,dy, \quad x\in\mathbb{R}, \tag{3.4}$$

where $w_k = 0$, if k is odd, and $w_k = -2^{\frac{k}{2}}$, when k is even.

Proof. We first observe that by Proposition 3.1 and since $\left(\frac{d}{dx}+x\right)^k = \sum_{\left(\substack{\rho,\sigma\in\mathbb{N}\\ 0\le\rho+\sigma\le k}\right)} c_{\rho,\sigma}x^\rho\frac{d^\sigma}{dx^\sigma}$, for certain $c_{\rho,\sigma}\in\mathbb{R}$, $\rho,\sigma\in\mathbb{N}$, $0\le\rho+\sigma\le k$, it is sufficient to prove that

$$\frac{d^k}{dx^k}H^{-\frac{k}{2}}\phi(x) = w_k\phi(x) + \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\phi(y)\frac{1}{\Gamma(\frac{k}{2})}\int_0^\infty t^{\frac{k}{2}-1}\frac{d^k}{dx^k}W_t(x,y)dtdy, \quad x\in\mathbb{R}, \tag{3.5}$$

where $w_k = 0$, if k is odd, and $w_k = -2^{\frac{k}{2}}$, when k is even.

By making the change of variable $t = \log\dfrac{1+s}{1-s}$ and by using (2.4), we can write, for every $s\in(0,1)$ and $x,y\in\mathbb{R}$,

$$\begin{aligned}
&\frac{d^k}{dx^k}W_{\log\frac{1+s}{1-s}}(x,y) = \left(\frac{1-s^2}{4\pi s}\right)^{\frac12}\frac{d^k}{dx^k}\left[e^{-\frac14\left(s(x+y)^2+\frac1s(x-y)^2\right)}\right]\\
&= \left(\frac{1-s^2}{4\pi s}\right)^{\frac12}\sum_{j=0}^k\binom{k}{j}\frac{d^j}{dx^j}\left(e^{-\frac{s}{4}(x+y)^2}\right)\frac{d^{k-j}}{dx^{k-j}}\left(e^{-\frac{1}{4s}(x-y)^2}\right)\\
&= \left(\frac{1-s^2}{4\pi s}\right)^{\frac12}e^{-\frac{s}{4}(x+y)^2}\frac{d^k}{dx^k}\left(e^{-\frac{1}{4s}(x-y)^2}\right)\\
&+ \left(\frac{1-s^2}{4\pi s}\right)^{\frac12}\sum_{j=1}^k\binom{k}{j}\left(\sum_{l=0}^{E[\frac{j}{2}]}E_{j,l}(x+y)^{j-2l}\Big(-\frac{s}{4}\Big)^{j-l}e^{-\frac{s}{4}(x+y)^2}\right)\\
&\times\left(\sum_{m=0}^{E[\frac{k-j}{2}]}E_{k-j,m}(x-y)^{k-j-2m}\Big(-\frac{1}{4s}\Big)^{k-j-m}e^{-\frac{1}{4s}(x-y)^2}\right).
\end{aligned}$$

On the other hand, by proceeding as in the proof of Proposition 3.1 it follows that

$$\begin{aligned}
&\left|\int_{\frac12}^1\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\frac{d^k}{dx^k}W_{\log\frac{1+s}{1-s}}(x,y)\frac{2}{1-s^2}ds\right|\\
&\le C\int_{\frac12}^1(-\log(1-s))^{\frac{k}{2}-1}\frac{1}{\sqrt{1-s}}ds \le C, \quad x,y\in\mathbb{R}.
\end{aligned} \tag{3.6}$$

We define, for every $j=1,...,k$, $0\leq l\leq E\left[\frac{j}{2}\right]$ and $0\leq m\leq E\left[\frac{k-j}{2}\right]$,

$$F^j_{l,m}(x,y)=(x+y)^{j-2l}(x-y)^{k-j-2m}$$
$$\times\int_0^{\frac{1}{2}}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\frac{e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}}{s^{k-2j+l-m+\frac{1}{2}}}\frac{1}{\sqrt{1-s^2}}ds,\quad x,y\in\mathbb{R}.$$

Since $\log\frac{1+s}{1-s}\sim s$, as $s\to 0^+$, and $j\geq 1$, it follows that

$$\begin{aligned}|F^j_{l,m}(x,y)| &\leq C|x+y|^{j-2l}|x-y|^{k-j-2m}\int_0^{\frac{1}{2}}s^{-\frac{k}{2}-\frac{3}{2}+2j-l+m}e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\,ds\\ &\leq C\int_0^{\frac{1}{2}}s^{j-\frac{3}{2}}e^{-c\frac{(x-y)^2}{s}}\,ds\leq C\int_0^{\frac{1}{2}}s^{-\frac{1}{2}}\,ds\leq C,\quad x,y\in\mathbb{R}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\int_0^{\frac{1}{2}}\left[\frac{d^k}{dx^k}W_{\log\frac{1+s}{1-s}}(x,y)-\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}}e^{-\frac{s}{4}(x+y)^2}\frac{d^k}{dx^k}\left(e^{-\frac{1}{4s}(x-y)^2}\right)\right]\right.\\ &\quad\left.\times\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\frac{2}{1-s^2}ds\right|\leq C,\quad x,y\in\mathbb{R}.\end{aligned}\tag{3.7}$$

We now observe that mean value theorem leads to

$$\begin{aligned}&\left|e^{-\frac{s}{4}(x+y)^2}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\frac{1}{\sqrt{1-s^2}}-(2s)^{\frac{k}{2}-1}\right|\leq\left|\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}-(2s)^{\frac{k}{2}-1}\right|\\ &+\left[\left|\frac{1}{\sqrt{1-s^2}}-1\right|e^{-\frac{s}{4}(x+y)^2}+\left|e^{-\frac{s}{4}(x+y)^2}-1\right|\right]\left|\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\right|\\ &\leq C\left(s^{\frac{k}{2}+1}+\left(s^2e^{-\frac{s}{4}(x+y)^2}+(x+y)^2s\right)\left|\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\right|\right),\end{aligned}$$

for every $s\in(0,\frac{1}{2})$ and $x,y\in\mathbb{R}$.

Then, since $\log \frac{1+s}{1-s} \sim s$, as $s \to 0^+$, by using again (2.4) we get

$$\begin{aligned}
&\left|\int_0^{\frac{1}{2}} \left(\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}} e^{-\frac{s}{4}(x+y)^2} \left(\log \frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \frac{2}{1-s^2} - \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}\right) \frac{d^k}{dx^k}\left(e^{-\frac{(x-y)^2}{4s}}\right) ds\right| \\
&\quad \leq C \int_0^{\frac{1}{2}} s^{\frac{k}{2}-\frac{1}{2}}(s+(x+y)^2) \left|\frac{d^k}{dx^k}\left(e^{-\frac{(x-y)^2}{4s}}\right)\right| ds \\
&\quad \leq C \sum_{n=0}^{E[\frac{k}{2}]} |x-y|^{k-2n} \int_0^{\frac{1}{2}} s^{-\frac{k}{2}-\frac{1}{2}+n}(s+(x+y)^2) e^{-\frac{(x-y)^2}{4s}} ds \\
&\quad \leq C(1+(x+y)^2), \quad x, y \in \mathbb{R}.
\end{aligned} \tag{3.8}$$

In view of properties (3.6), (3.7) and (3.8), to establish the desired property (3.5) it is sufficient to prove that

$$\begin{aligned}
&\frac{d^k}{dx^k} \int_{-\infty}^{+\infty} \phi(y) \frac{1}{\Gamma(\frac{k}{2})} \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} e^{-\frac{1}{4s}(x-y)^2} ds dy \\
&\qquad = w_k \phi(x) + \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \frac{1}{\Gamma(\frac{k}{2})} \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} \frac{d^k}{dx^k}\left(e^{-\frac{1}{4s}(x-y)^2}\right) ds dy, \ x \in \mathbb{R},
\end{aligned}$$

where $w_k = 0$, if k is odd, and $w_k = -2^{\frac{k}{2}}$, when k is even.

As earlier we can see that, for each $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
&\frac{d^{k-1}}{dx^{k-1}} \int_{-\infty}^{+\infty} \phi(y) \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} e^{-\frac{1}{4s}(x-y)^2} ds dy \\
&\qquad = \int_{-\infty}^{+\infty} \phi(y) \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} \frac{d^{k-1}}{dx^{k-1}}\left(e^{-\frac{1}{4s}(x-y)^2}\right) ds dy, \quad x \in \mathbb{R}.
\end{aligned}$$

Let us represent by Φ the following function

$$\Phi(x) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} \frac{d^{k-1}}{dx^{k-1}}\left(e^{-\frac{x^2}{4s}}\right) ds, \quad x \in \mathbb{R}.$$

By proceeding as above we can see that $\Phi \in L^1(\mathbb{R})$. Indeed, (2.4) leads to

$$\Phi(x) = \frac{1}{\Gamma(\frac{k}{2})} \sum_{l=0}^{E[\frac{k-1}{2}]} (-1)^{k-1-l} E_{k-1,l} x^{k-1-2l} \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} \left(\frac{1}{4s}\right)^{k-1-l} e^{-\frac{x^2}{4s}} ds, \quad x \in \mathbb{R}.$$

and then, according to [30, Lemma 1.1],

(3.9)

$$|\Phi(x)| \leq C \sum_{l=0}^{E[\frac{k-1}{2}]} |x|^{k-1-2l} \int_0^{\frac{1}{2}} s^{-\frac{k}{2}+l-\frac{1}{2}} e^{-\frac{x^2}{4s}} ds \leq Ce^{-\frac{x^2}{4}} \int_0^{\frac{1}{2}} \frac{e^{-c\frac{x^2}{s}}}{s} ds \leq C \frac{e^{-\frac{x^2}{4}}}{\sqrt{|x|}}, \quad x \in \mathbb{R}.$$

Hence $\Phi \in L^1(\mathbb{R})$. Moreover $\Phi \in C^\infty(\mathbb{R} \setminus \{0\})$.

Also, when k is even, we can see that

$$\lim_{\varepsilon \to 0} \Phi(\varepsilon) = -2^{\frac{k}{2}-1}. \tag{3.10}$$

In effect, if k is even we can write

$$\Phi(\varepsilon) = -\frac{1}{\Gamma(\frac{k}{2})\sqrt{\pi}} \sum_{l=0}^{\frac{k}{2}-1} (-1)^l E_{k-1,l} \frac{\varepsilon^{k-1-2l}}{2^{\frac{3k}{2}-2l-1}} \int_0^{\frac{1}{2}} \frac{e^{-\frac{\varepsilon^2}{4s}}}{s^{\frac{k}{2}+\frac{1}{2}-l}} ds, \quad \varepsilon \in \mathbb{R}.$$

Hence, the duplication formula ([18, (1.2.3)]) allows us to write

$$\begin{aligned}
\lim_{\varepsilon \to 0} \Phi(\varepsilon) &= -\frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})\sqrt{\pi}} \lim_{\varepsilon \to 0} \sum_{l=0}^{\frac{k}{2}-1} (-1)^l E_{k-1,l} \int_{\frac{\varepsilon^2}{2}}^{\infty} e^{-u} u^{\frac{k}{2}-\frac{3}{2}-l} du \\
&= \frac{-1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})\sqrt{\pi}} \sum_{l=0}^{\frac{k}{2}-1} (-1)^l E_{k-1,l} \Gamma\Big(\frac{k-1}{2} - l\Big) = \frac{-(k-1)!}{2^{\frac{k}{2}-1}(\Gamma(\frac{k}{2}))^2} \sum_{l=0}^{\frac{k}{2}-1} (-1)^l \binom{\frac{k}{2}-1}{l} \frac{1}{k-1-2l} \\
&= \frac{-(k-1)!}{2^{\frac{k}{2}-1}(\Gamma(\frac{k}{2}))^2} \int_0^1 (1-t^2)^{\frac{k}{2}-1} dt = \frac{-(k-1)!}{2^{\frac{k}{2}}(\Gamma(\frac{k}{2}))^2} \frac{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2})} = -2^{\frac{k}{2}-1},
\end{aligned}$$

and (3.10) is thus established.

For every $x \in \mathbb{R}$, we can write

$$\begin{aligned}
\frac{d}{dx} \int_{-\infty}^{+\infty} \phi(y)\Phi(x-y) dy &= \frac{d}{dx} \int_{-\infty}^{+\infty} \phi(x-y)\Phi(y) dy = -\int_{-\infty}^{+\infty} \frac{d}{dy}(\phi(x-y))\Phi(y) dy \\
&= -\lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{d}{dy}(\phi(x-y))\Phi(y) dy \\
&= \lim_{\varepsilon \to 0^+} \left[\int_{|y|>\varepsilon} \phi(x-y) \frac{d}{dy}\Phi(y) dy - \phi(x+\varepsilon)\Phi(-\varepsilon) + \phi(x-\varepsilon)\Phi(\varepsilon) \right] \\
&= \lim_{\varepsilon \to 0^+} \left[\int_{|x-y|>\varepsilon} \phi(y) \Big(\frac{d}{dy}\Phi\Big)(x-y) dy + \phi(x-\varepsilon)\Phi(\varepsilon) - \phi(x+\varepsilon)\Phi(-\varepsilon) \right].
\end{aligned}$$

Suppose now that k is odd. Then Φ is an even function and from (3.9) we obtain, for every $x \in \mathbb{R}$,

$$|\phi(x-\varepsilon)\Phi(\varepsilon) - \phi(x+\varepsilon)\Phi(-\varepsilon)| \leq C\varepsilon|\Phi(\varepsilon)| \longrightarrow 0, \qquad \text{as } \varepsilon \to 0^+ .$$

On the other hand, assuming that k is even, (3.10) leads to

$$\lim_{\varepsilon\to 0^+} \phi(x-\varepsilon)\Phi(\varepsilon) - \phi(x+\varepsilon)\Phi(-\varepsilon) = \lim_{\varepsilon\to 0^+} (\phi(x+\varepsilon) + \phi(x-\varepsilon))\Phi(\varepsilon) = -2^{\frac{k}{2}}\phi(x),$$

for every $x \in \mathbb{R}$.

Hence,

$$\frac{d}{dx}\int_{-\infty}^{+\infty} \phi(y)\Phi(x-y)dy = w_k\phi(x) + \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} \phi(y)\frac{d}{dx}(\Phi(x-y))dy, \quad x \in \mathbb{R},$$

where $w_k = 0$, if k is odd, and $w_k = -2^{\frac{k}{2}}$, when k is even. Thus the proof is finished.

□

The following relation between the kernels $R_\alpha^{(k)}(x,y)$ and $R^{(k)}(x,y)$, $x, y \in (0,\infty)$, is the key of our procedure in order to establish that the k-order Riesz transform associated with the Laguerre operator is a principal value integral operator.

Proposition 3.11. *Let $\alpha > -1$ and $k \in \mathbb{N}$. We have that*

(i) $|R_\alpha^{(k)}(x,y)| \leq C\dfrac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}$, $0 < y < \dfrac{x}{2}$.

(ii) $|R_\alpha^{(k)}(x,y)| \leq C\dfrac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}}$, $y > 2x$ and k even, and $|R_\alpha^{(k)}(x,y)| \leq C\dfrac{x^{\alpha+\frac{3}{2}}}{y^{\alpha+\frac{5}{2}}}$, $y > 2x$ and k odd.

(iii) $\left|R_\alpha^{(k)}(x,y) - R^{(k)}(x,y)\right| \leq C\dfrac{1}{x}\left(1 + \left(\dfrac{x}{|x-y|}\right)^{\frac{1}{2}}\right)$, $\dfrac{x}{2} < y < 2x$.

Proof. We first establish the following formula that will be used later. For every $j \in \mathbb{N}$, and $t, x, y \in (0,\infty)$ we have that

$$\begin{aligned}&\frac{d^j}{dx^j}\left[\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\right]\\ &= \sum_{n=0}^{E[\frac{j}{2}]} E_{j,n}\frac{x^{j-2n}}{2^{j-n}}\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)^{2(j-n)}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha+n-j} I_{\alpha-n+j}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right).\end{aligned} \tag{3.12}$$

Indeed, let $j \in \mathbb{N}$. By using (2.4) and since $\frac{d}{dx}g(x) = \frac{1}{2}\left(\frac{1}{u}\frac{d}{du}\right)[g(u^2)]_{|u=\sqrt{x}}$ we can write

$$\begin{aligned}
&\frac{d^j}{dx^j}\left[\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\right] \\
&= \sum_{n=0}^{E[\frac{j}{2}]} E_{j,n} x^{j-2n} \frac{d^{j-n}}{dz^{j-n}}\left(\left(\frac{2\sqrt{z}ye^{-t}}{1-e^{-2t}}\right)^{-\alpha} I_\alpha\left(\frac{2\sqrt{z}ye^{-t}}{1-e^{-2t}}\right)\right)_{\Big|z=x^2} \\
&= \sum_{n=0}^{E[\frac{j}{2}]} E_{j,n} \frac{x^{j-2n}}{2^{j-n}}\Big(\frac{1}{x}\frac{d}{dx}\Big)^{j-n}\left(\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\right), \quad t,x,y \in (0,\infty).
\end{aligned}$$

Thus, (3.12) can be easily deduced from property (P3).

Let us now prove (i) and (ii). Since $\frac{d}{dx} + x = e^{-\frac{x^2}{2}}\frac{d}{dx}e^{\frac{x^2}{2}}$ we can write

$$\begin{aligned}
\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= x^{\alpha+\frac{1}{2}} e^{-\frac{x^2}{2}} \frac{d^k}{dx^k}\Big(e^{\frac{x^2}{2}} x^{-\alpha-\frac{1}{2}} W_t^\alpha(x,y)\Big) \\
&= \left(\frac{2e^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} \left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\alpha+\frac{1}{2}} e^{-\frac{x^2}{2}-\frac{y^2}{2}\frac{1+e^{-2t}}{1-e^{-2t}}} \\
&\times \frac{d^k}{dx^k}\left[\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-x^2\frac{e^{-2t}}{1-e^{-2t}}}\right], \quad t,x,y \in (0,\infty).
\end{aligned}$$

By taking into account formulas (2.4) and (3.12) we get

$$\begin{aligned}
&\frac{d^k}{dx^k}\left[\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-x^2\frac{e^{-2t}}{1-e^{-2t}}}\right] \\
&= \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{dx^j}\left[\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)^{-\alpha} I_\alpha\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)\right] \frac{d^{k-j}}{dx^{k-j}}\left[e^{-x^2\frac{e^{-2t}}{1-e^{-2t}}}\right] \\
&= e^{-x^2\frac{e^{-2t}}{1-e^{-2t}}} \sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} \binom{k}{j} \frac{E_{j,n}E_{k-j,m}}{2^{j-n}} \Big(\frac{2ye^{-t}}{1-e^{-2t}}\Big)^{2(j-n)} \Big(\frac{-e^{-2t}}{1-e^{-2t}}\Big)^{k-j-m} \\
&\times x^{k-2m-2n}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)^{-\alpha-j+n} I_{\alpha+j-n}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big), \quad t,x,y \in (0,\infty).
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= \Big(\frac{2e^{-t}}{1-e^{-2t}}\Big)^{\frac{1}{2}}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}} \\ &\times\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]}\binom{k}{j}\frac{E_{j,n}E_{k-j,m}}{2^{j-n}}\Big(\frac{2ye^{-t}}{1-e^{-2t}}\Big)^{2(j-n)}\Big(\frac{-e^{-2t}}{1-e^{-2t}}\Big)^{k-j-m} \\ &\times x^{k-2m-2n}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)^{-\alpha-j+n} I_{\alpha+j-n}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big),\quad t,x,y\in(0,\infty).\end{aligned} \tag{3.13}$$

By using property (P1) it follows that

$$\begin{aligned}&\Big|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\leq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\Big| \leq C(xy)^{\alpha+\frac{1}{2}}\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} x^{k-2m-2n}y^{2(j-n)} \\ &\times\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\leq 1}^{\infty} t^{\frac{k}{2}-1}e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}}\Big(\frac{e^{-t}}{1-e^{-2t}}\Big)^{\alpha+1+j+k-2n-m}dt \\ &\leq C(xy)^{\alpha+\frac{1}{2}}\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} x^{k-2m-2n}y^{2(j-n)} \\ &\times\Big(\int_0^1 t^{-\frac{k}{2}-\alpha-2-j+2n+m}e^{-c\frac{x^2+y^2}{t}}dt + e^{-c(x^2+y^2)}\int_1^\infty t^{\frac{k}{2}-1}e^{-(\alpha+1)t}dt\Big)\end{aligned}$$

Hence by taking into account [30, Lemma 1.1] we conclude that

$$\begin{aligned}\Big|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\leq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\Big| &\leq C\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]}\frac{(xy)^{\alpha+\frac{1}{2}}x^{k-2m-2n}y^{2(j-n)}}{(x^2+y^2)^{\frac{k}{2}+\alpha+1+j-2n-m}} \\ &\leq C\frac{(xy)^{\alpha+\frac{1}{2}}}{(x^2+y^2)^{\alpha+1}} \leq C\begin{cases}\dfrac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, & 0<y<x\ , \\[2ex] \dfrac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}}, & y>x>0.\end{cases}\end{aligned} \tag{3.14}$$

Note that if k is odd we can improve the estimate when $y>x>0$ as follows

$$\Big|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\leq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\Big| \leq C\frac{(xy)^{\alpha+\frac{1}{2}}x}{(x^2+y^2)^{\alpha+\frac{3}{2}}} \leq \frac{x^{\alpha+\frac{3}{2}}}{y^{\alpha+\frac{5}{2}}}. \tag{3.15}$$

Assume now that $\frac{2xye^{-t}}{1-e^{-2t}} \geq 1$. From (3.13) and property (P2) we get

$$\begin{aligned}\left|\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\right| &\leq C \sum_{j=0}^{k} \sum_{n=0}^{E[\frac{j}{2}]} \sum_{m=0}^{E[\frac{k-j}{2}]} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}} + \frac{2xye^{-t}}{1-e^{-2t}}} \\ &\quad \times x^{k-2m-2n} y^{2(j-n)} \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{k-m-2n+j+\frac{1}{2}}, \quad t,x,y \in (0,\infty).\end{aligned}$$

We also observe that

$$-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}} + \frac{2xye^{-t}}{1-e^{-2t}} = -\frac{(x-ye^{-t})^2+(y-xe^{-t})^2}{2(1-e^{-2t})}.$$

Thus, if $0<y<\frac{x}{2}$, we can write

$$\begin{aligned}\left|\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\right| &\leq C \sum_{j=0}^{k} \sum_{n=0}^{E[\frac{j}{2}]} \sum_{m=0}^{E[\frac{k-j}{2}]} e^{-\frac{x^2}{8(1-e^{-2t})}} x^{k-2m-2n+2(j-n)} \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{k-m-2n+j+\frac{1}{2}} \\ &\leq C e^{-c\frac{x^2}{1-e^{-2t}}} \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\frac{k}{2}+\frac{1}{2}}, \quad t \in (0,\infty).\end{aligned}$$

Hence, if $-1<\alpha<-\frac{1}{2}$, [30, Lemma 1.1] leads to

$$\left|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\right| \leq C\left(\int_0^1 \frac{e^{-c\frac{x^2}{t}}}{t^{\frac{3}{2}}}dt + e^{-cx^2}\right) \leq C\frac{1}{x} \leq C\frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, \ \ 0<y<\frac{x}{2}.$$

For $\alpha>-\frac{1}{2}$ we can proceed as follows.

$$\begin{aligned}&\left|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\right| \leq C(xy)^{\alpha+\frac{1}{2}} \int_0^\infty t^{\frac{k}{2}-1} e^{-c\frac{x^2}{1-e^{-2t}}} \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\frac{k}{2}+\alpha+1} dt \\ &\leq C(xy)^{\alpha+\frac{1}{2}}\left(\int_0^1 \frac{e^{-c\frac{x^2}{t}}}{t^{\alpha+2}}dt + e^{-cx^2}\right) \leq C\frac{(xy)^{\alpha+\frac{1}{2}}}{x^{2\alpha+2}} \leq C\frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, \quad 0<y<\frac{x}{2}.\end{aligned}$$

In a similar way, if $0<2x<y$, we can write

$$\begin{aligned}&\left|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\right| \leq C(xy)^{\alpha+\frac{3}{2}} \int_0^\infty t^{\frac{k}{2}-1} e^{-c\frac{y^2}{1-e^{-2t}}} \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\frac{k}{2}+\alpha+2} dt \\ &\leq C(xy)^{\alpha+\frac{3}{2}}\left(\int_0^1 \frac{e^{-c\frac{y^2}{t}}}{t^{\alpha+3}}dt + e^{-cy^2}\right) \leq C\frac{(xy)^{\alpha+\frac{3}{2}}}{y^{2\alpha+4}} \leq C\frac{x^{\alpha+\frac{3}{2}}}{y^{\alpha+\frac{5}{2}}}.\end{aligned}$$

These estimations allow us to get

$$
\left|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)dt\right| \leq C\begin{cases} \dfrac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, & 0<y<\dfrac{x}{2}\ , \\[2ex] \dfrac{x^{\alpha+\frac{3}{2}}}{y^{\alpha+\frac{5}{2}}}, & y>2x>0. \end{cases} \tag{3.16}
$$

Hence, by (3.14), (3.15) and (3.16), (i) and (ii) are proved.

Next we establish statement (iii). Observe first that, since $\frac{d}{dx}+x=e^{-\frac{x^2}{2}}\frac{d}{dx}e^{\frac{x^2}{2}}$,

$$
\begin{aligned}
&\mathfrak{D}_\alpha^k W_t^\alpha(x,y)=x^{\alpha+\frac{1}{2}}\Big(\frac{d}{dx}+x\Big)^k[x^{-\alpha-\frac{1}{2}}W_t^\alpha(x,y)]\\
&=\sqrt{2\pi}x^{\alpha+\frac{1}{2}}\Big(\frac{d}{dx}+x\Big)^k\left[x^{-\alpha-\frac{1}{2}}e^{\frac{-2xye^{-t}}{1-e^{-2t}}}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}}I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)W_t(x,y)\right]\\
&=\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\alpha+\frac{1}{2}}\sum_{j=0}^{k}\binom{k}{j}\frac{d^j}{dx^j}\left[e^{\frac{-2xye^{-t}}{1-e^{-2t}}}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha}I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\right]\Big(\frac{d}{dx}+x\Big)^{k-j}W_t(x,y)\\
&=\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\alpha+\frac{1}{2}}\sum_{j=0}^{k}\binom{k}{j}\Big(\frac{d}{dx}+x\Big)^{k-j}W_t(x,y)\\
&\times\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)^{j-l}e^{-\frac{2xye^{-t}}{1-e^{-2t}}}\frac{d^l}{dx^l}\left(\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-\alpha}I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\right),\quad t,x,y\in(0,\infty).
\end{aligned}
$$

Hence, by using formula (3.12) we obtain that, for every $t,x,y\in(0,\infty)$,

$$
\begin{aligned}
\mathfrak{D}_\alpha^k W_t^\alpha(x,y)&=\sqrt{2\pi}e^{-\frac{2xye^{-t}}{1-e^{-2t}}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\Big(\frac{d}{dx}+x\Big)^{k-j}(W_t(x,y))\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)^j\\
&\times\sum_{n=0}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}(-1)^l\binom{j}{l}\frac{E_{l,n}}{2^{l-n}}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{-n}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}}I_{\alpha-n+l}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right).
\end{aligned}
\tag{3.17}
$$

Let us consider now $x, y, t \in (0, \infty)$ such that $\frac{2xye^{-t}}{1-e^{-2t}} \geq 1$. By taking into account property (P2) and (3.17) we can write

$$\mathfrak{D}_\alpha^k W_t^\alpha(x,y) = \Big(\frac{d}{dx}+x\Big)^k (W_t(x,y)) \left(1 + O\left(\frac{1-e^{-2t}}{xye^{-t}}\right)\right)$$
$$+ \sum_{j=1}^k (-1)^j \binom{k}{j} \Big(\frac{d}{dx}+x\Big)^{k-j} (W_t(x,y)) \left(\frac{2ye^{-t}}{1-e^{-2t}}\right)^j \sum_{n=0}^{E[\frac{j}{2}]} \sum_{l=2n}^{j} (-1)^l \binom{j}{l} \frac{E_{l,n}}{2^{l-n}} \left(\frac{1-e^{-2t}}{2xye^{-t}}\right)^n$$
$$\times \left(\sum_{r=0}^{E[\frac{j}{2}]} \frac{(-1)^r [\alpha + l - n, r]}{2^r} \left(\frac{1-e^{-2t}}{2xye^{-t}}\right)^r + O\left(\left(\frac{1-e^{-2t}}{xye^{-t}}\right)^{E[\frac{j}{2}]+1}\right)\right)$$
$$= \Big(\frac{d}{dx}+x\Big)^k (W_t(x,y)) + \sum_{j=1}^k (-1)^j \binom{k}{j} \Big(\frac{d}{dx}+x\Big)^{k-j} (W_t(x,y)) \left(\frac{2ye^{-t}}{1-e^{-2t}}\right)^j$$
$$\times \sum_{n=0}^{E[\frac{j}{2}]} \sum_{l=2n}^{j} \sum_{r=0}^{E[\frac{j}{2}]} (-1)^{l+r} \binom{j}{l} \frac{E_{l,n}}{2^{l-n}} \frac{[\alpha + l - n, r]}{2^r} \left(\frac{1-e^{-2t}}{2xye^{-t}}\right)^{n+r}$$
$$+ \sum_{j=0}^k (-1)^j \binom{k}{j} \Big(\frac{d}{dx}+x\Big)^{k-j} (W_t(x,y)) O\left(\left(\frac{ye^{-t}}{1-e^{-2t}}\right)^{j-E[\frac{j}{2}]-1} \frac{1}{x^{E[\frac{j}{2}]+1}}\right).$$

Lemma 2.5 allows us to see that, for every $j \in \mathbb{N}$, $j = 1, ..., k$,

$$\sum_{n=0}^{E[\frac{j}{2}]} \sum_{l=2n}^{j} \sum_{r=0}^{E[\frac{j}{2}]} (-1)^{l+r} \binom{j}{l} \frac{E_{l,n}}{2^{l-n}} \frac{[\alpha + l - n, r]}{2^r} \left(\frac{1-e^{-2t}}{2xye^{-t}}\right)^{n+r}$$
$$= \sum_{n=0}^{E[\frac{j}{2}]} \sum_{l=2n}^{j} \sum_{m=n}^{E[\frac{j}{2}]+n} (-1)^{l+m-n} \binom{j}{l} \frac{E_{l,n}}{2^{l-n}} \frac{[\alpha + l - n, m-n]}{2^{m-n}} \left(\frac{1-e^{-2t}}{2xye^{-t}}\right)^{m}$$
$$= \sum_{m=0}^{E[\frac{j}{2}]} \left(\frac{1-e^{-2t}}{4xye^{-t}}\right)^m \sum_{n=0}^{m} \sum_{l=2n}^{j} (-1)^{l+m-n} \binom{j}{l} \frac{E_{l,n}}{2^{l-2n}} [\alpha + l - n, m-n]$$
$$+ \sum_{m=E[\frac{j}{2}]+1}^{2E[\frac{j}{2}]} \left(\frac{1-e^{-2t}}{4xye^{-t}}\right)^m \sum_{n=m-E[\frac{j}{2}]}^{E[\frac{j}{2}]} \sum_{l=2n}^{j} (-1)^{l+m-n} \binom{j}{l} \frac{E_{l,n}}{2^{l-2n}} [\alpha + l - n, m-n]$$
$$= O\left(\left(\frac{1-e^{-2t}}{xye^{-t}}\right)^{E[\frac{j}{2}]+1}\right).$$

Hence, it follows that

$$\begin{aligned}\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= \Big(\frac{d}{dx}+x\Big)^k W_t(x,y)\\ &\quad+\sum_{j=0}^k(-1)^j\binom{k}{j}\Big(\frac{d}{dx}+x\Big)^{k-j}(W_t(x,y))O\left(\left(\frac{ye^{-t}}{1-e^{-2t}}\right)^{j-E[\frac{j}{2}]-1}\frac{1}{x^{E[\frac{j}{2}]+1}}\right).\end{aligned}$$

Assume that $0<\frac{x}{2}<y<2x$. In order to establish (iii) we now proceed as in the proof of Proposition 3.1. First note that by formula (3.2)

$$\begin{aligned}&\left|\mathfrak{D}_\alpha^k W_t^\alpha(x,y)-\Big(\frac{d}{dx}+x\Big)^k W_t(x,y)\right|\\ &\leq C\sum_{j=0}^k\sum_{0\leq\rho+\sigma\leq k-j}x^\rho\left|\frac{d^\sigma}{dx^\sigma}W_t(x,y)\right|\left(\frac{ye^{-t}}{1-e^{-2t}}\right)^{j-E[\frac{j}{2}]-1}\frac{1}{x^{E[\frac{j}{2}]+1}}.\end{aligned}$$

Assume that $j,\rho,\sigma,b_1,b_2\in\mathbb{N}$, $0\leq j\leq k$, $0\leq\rho+\sigma\leq k-j$ and $2b_1+b_2\leq\sigma$. According to [30, p. 50] and by making the change of variable $t=\log\frac{1+s}{1-s}$, we must analyze the following integral.

$$\begin{aligned}I_{\rho,\sigma,j}^{b_1,b_2}(x,y) &= \frac{x^\rho y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^1\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\left(\frac{1-s^2}{s}\right)^{j-E[\frac{j}{2}]-\frac{1}{2}}\\ &\quad\times\left(s+\frac{1}{s}\right)^{b_1}e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\left(s(x+y)+\frac{1}{s}(x-y)\right)^{b_2}\frac{ds}{1-s^2}\\ &= J_{\rho,\sigma,j}^{b_1,b_2}(x,y)+H_{\rho,\sigma,j}^{b_1,b_2}(x,y),\quad x,y\in(0,\infty),\end{aligned}$$

where J and H are defined as I but replacing the integral over $(0,1)$ by the integral over $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$, respectively.

Since $\log\frac{1+s}{1-s}\sim s$, as $s\to 0^+$, it follows that

$$\begin{aligned}J_{\rho,\sigma,j}^{b_1,b_2}(x,y) &\leq C\frac{x^\rho y^j}{(xy)^{1+E[\frac{j}{2}]}}\\ &\quad\times\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}}s^{\frac{k}{2}-\frac{1}{2}-j+E[\frac{j}{2}]-b_1}e^{-\frac{1}{4}(s(x+y)^2+\frac{(x-y)^2}{s})}\left|s(x+y)+\frac{(x-y)}{s}\right|^{b_2}ds\\ &\leq C\frac{y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}}s^{\frac{1}{2}(k-j-2b_1-b_2-\rho)+E[\frac{j}{2}]-\frac{j}{2}-\frac{1}{2}}e^{-c\frac{(x-y)^2}{s}}ds\end{aligned}$$

$$\leq C\frac{y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}} s^{E[\frac{j}{2}]-\frac{j}{2}-\frac{1}{2}}e^{-c\frac{(x-y)^2}{s}}ds.$$

By taking into account that $0<\frac{x}{2}<y<2x$ and using [30, Lemma 1.1] we get

$$\begin{aligned} J_{\rho,\sigma,j}^{b_1,b_2}(x,y) &\leq C\frac{x^{j-2E[\frac{j}{2}]-\frac{3}{2}}}{\sqrt{x}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}} s^{E[\frac{j}{2}]-\frac{j}{2}-\frac{1}{2}}e^{-c\frac{(x-y)^2}{s}}ds \\ &\leq C\frac{1}{\sqrt{x}}\int_0^{\frac{1}{2}}\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{\frac{5}{4}}}ds \leq C\frac{1}{x}\left(\frac{x}{|x-y|}\right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, since that $\log\frac{1+s}{1-s}\sim -\log(1-s)$, as $s\to 1^-$, we have that

$$\begin{aligned} H_{\rho,\sigma,j}^{b_1,b_2}(x,y) &\leq C\frac{x^\rho y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{\frac{1}{2},\frac{(1-s^2)xy}{2s}\geq 1}^{1}(-\log(1-s))^{\frac{k}{2}-1}(1-s)^{j-E[\frac{j}{2}]-\frac{3}{2}}e^{-cs(x+y)^2}ds \\ &\leq Ce^{-c(x+y)^2}\int_{\frac{1}{2}}^{1}(-\log(1-s))^{\frac{k}{2}-1}(1-s)^{j-\frac{1}{2}}ds \leq Ce^{-c(x+y)^2}, \quad x,y\in(0,\infty). \end{aligned}$$

Hence we conclude that, if $0<\frac{x}{2}<y<2x$,

$$\left|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\geq 1}^{\infty} t^{\frac{k}{2}-1}\Big(\mathfrak{D}_\alpha^k W_t^\alpha(x,y)-\Big(\frac{d}{dx}+x\Big)^k W_t(x,y)\Big)dt\right| \leq C\frac{1}{x}\left(\frac{x}{|x-y|}\right)^{\frac{1}{2}}. \tag{3.18}$$

Also, by using again (2.4) we obtain, for each $t,x,y\in(0,\infty)$,

$$\begin{aligned} \Big(\frac{d}{dx}+x\Big)^k W_t(x,y) &= e^{-\frac{x^2}{2}}\frac{d^k}{dx^k}\left[e^{\frac{x^2}{2}}W_t(x,y)\right] \\ &= W_t(x,y)\sum_{j=0}^{k}\sum_{l=0}^{E[\frac{j}{2}]}\binom{k}{j}E_{j,l}x^{j-2l}\Big(\frac{2ye^{-t}}{1-e^{-2t}}\Big)^{k-j}\Big(\frac{-e^{-2t}}{1-e^{-2t}}\Big)^{j-l}. \end{aligned}$$

Hence it follows that, when $0<\frac{x}{2}<y<2x$,

$$\begin{aligned} &\left|\int_{0,\frac{2xye^{-t}}{1-e^{-2t}}\leq 1}^{\infty} t^{\frac{k}{2}-1}\Big(\frac{d}{dx}+x\Big)^k W_t(x,y)dt\right| \\ &\leq C\sum_{l=0}^{E[\frac{k}{2}]}x^{k-2l}\left(\int_0^1 t^{-\frac{k}{2}-\frac{3}{2}+l}e^{-c\frac{x^2}{t}}dt+e^{-cx^2}\int_1^\infty t^{\frac{k}{2}-1}e^{-\frac{t}{2}}dt\right)\leq C\frac{1}{x}. \end{aligned} \tag{3.19}$$

The estimations (3.14), (3.18) and (3.19) allow us to finish the proof of (iii). $\square$

By proceeding as above and having in mind Proposition 3.1 we can see that, for every $k\in\mathbb{N}$ and $\phi\in C_c^\infty(0,\infty)$, the function $L^{-\frac{k}{2}}\phi$ is $(k-1)$-times differentiable on $(0,\infty)$ and k-times

differentiable on $(0,\infty)\setminus \operatorname{supp}\phi$. Moreover,

$$\mathfrak{D}_\alpha^\ell L^{-\frac{k}{2}}\phi(x)=\frac{1}{\Gamma(\frac{k}{2})}\int_0^\infty \phi(y)\int_0^\infty t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^\ell W_t^\alpha(x,y)dtdy, \tag{3.20}$$

for every $x\in(0,\infty)$, when $\ell=0,1,\dots,k-1$, and for every $x\in(0,\infty)\setminus \operatorname{supp}\phi$, when $\ell=k$.

We now prove that, for every $\phi\in C_c^\infty(0,\infty)$ and $k\in\mathbb{N}$, $L_\alpha^{-\frac{k}{2}}\phi$ is k-times differentiable on $(0,\infty)$ and that $\mathfrak{D}_\alpha^k L^{-\frac{k}{2}}\phi$ is a principal value integral operator (modulus a constant times of the function when k is even).

Proposition 3.21. *Let $\alpha>-1$, $k\in\mathbb{N}$ and $\phi\in C_c^\infty(0,\infty)$. Then*

$$\mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}}\phi(x)=w_k\phi(x)+\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)\phi(y)\,dy,\quad x\in(0,\infty),$$

where $w_k=0$, when k is odd and $w_k=-2^{\frac{k}{2}}$, when k is even.

Proof. For every $x\in(0,\infty)$, Proposition 3.11 implies that

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)\phi(y)dy\\
&=\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty (R_\alpha^{(k)}(x,y)-R^{(k)}(x,y))\phi(y)dy+\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R^{(k)}(x,y)\phi(y)dy\\
&=\int_0^\infty\left(\mathfrak{D}_\alpha^k K_{\alpha,k}(x,y)-\left(\frac{d}{dx}+x\right)^k K_k(x,y)\right)\phi(y)dy+\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R^{(k)}(x,y)\phi(y)dy\\
&=\frac{d}{dx}\left(\int_0^\infty\left[\mathfrak{D}_\alpha^{k-1}K_{\alpha,k}(x,y)-\left(\frac{d}{dx}+x\right)^{k-1}K_k(x,y)\right]\phi(y)dy\right)\\
&+\left(x-\frac{\alpha+\frac12}{x}\right)\int_0^\infty\mathfrak{D}_\alpha^{k-1}(K_{\alpha,k}(x,y))\phi(y)dy-x\int_0^\infty\left(\frac{d}{dx}+x\right)^{k-1}(K_k(x,y))\phi(y)dy\\
&+\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R^{(k)}(x,y)\phi(y)dy.
\end{aligned}$$

By taking into account Propositions 3.1 and 3.3 and (3.20) we can conclude that

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)\phi(y)dy\\
&\quad=\frac{d}{dx}\left(\int_0^\infty\left[\mathfrak{D}_\alpha^{k-1}K_{\alpha,k}(x,y)-\left(\frac{d}{dx}+x\right)^{k-1}K_k(x,y)\right]\phi(y)dy\right)\\
&\quad+\left(x-\frac{\alpha+\frac12}{x}\right)\int_0^\infty\mathfrak{D}_\alpha^{k-1}K_{\alpha,k}(x,y)\phi(y)dy-w_k\phi(x)
\end{aligned}$$

$$+\frac{d}{dx}\int_0^\infty \Big(\frac{d}{dx}+x\Big)^{k-1} K_k(x,y)\phi(y)dy = \mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}}\phi(x) - w_k\phi(x), \quad x\in(0,\infty),$$

where $w_k = 0$, for k odd, and $w_k = -2^{\frac{k}{2}}$, when k is even. Thus the proof is finished. $\square$

We now prove the main result of the paper.

Proof of Theorem 1.3. We consider the maximal operator associated with $R_\alpha^{(k)}$ defined by

$$R_{\alpha,*}^{(k)}f(x) = \sup_{\varepsilon>0}\left|\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)f(y)dy\right|, \quad f\in C_c^\infty(0,\infty), x\in(0,\infty).$$

According to Proposition 3.11 we get

$$R_{\alpha,*}^{(k)}f(x) \le C(H_0^{\alpha+\frac{1}{2}}(|f|)(x) + H_\infty^{\alpha+\frac{1}{2}+\delta_k}(|f|)(x) + R_{\mathrm{loc},*}^{(k)}(f)(x) + N(f)(x)),$$

where $\delta_k = 1$, when k is odd, $\delta_k = 0$, when k is even,

$$R_{\mathrm{loc},*}^{(k)}(f)(x) = \sup_{\varepsilon>0}\left|\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\Big(\frac{d}{dx}+x\Big)^k K_k(x,y)f(y)dy\right|,$$

and

$$N(f)(x) = \int_{\frac{x}{2}}^{2x} f(y)\frac{1}{y}\left(1+\Big(\frac{x}{|x-y|}\Big)^{\frac{1}{2}}\right)dy.$$

By [5, Lemma 3.1] $H_0^{\alpha+\frac{1}{2}}$ is of strong type (p,p) with respect to $x^\delta dx$, when $1<p<\infty$ and $\delta < \Big(\alpha+\frac{3}{2}\Big)p-1$, and of weak type $(1,1)$ when $\delta\le\alpha+\frac{1}{2}$. Also from [5, Lemma 3.2] the operator $H_\infty^{\alpha+\frac{1}{2}+\delta_k}$ is of strong type (p,p) for $x^\delta dx$, when $1<p<\infty$ and $-\Big(\alpha+\frac{1}{2}\Big)p-1<\delta$, and of weak type $(1,1)$ with respect to $x^\delta dx$ when $-\alpha-\frac{5}{2}\le\delta$, if k is odd; and, in the case that k is even, when $\delta\ge-\alpha-\frac{3}{2}$, and $\alpha\ne-\frac{1}{2}$ and when $\delta>-1$ and $\alpha=-\frac{1}{2}$.

On the other hand, by using Jensen inequality we can see that the operator N is bounded from $L^p((0,\infty),x^\delta dx)$ into itself, for every $1\le p<\infty$ and $\delta\in\mathbb{R}$.

In [30] it was established that the kernel $R^{(k)}(x,y)$, $x,y\in\mathbb{R}$, is a Calderón-Zygmund kernel. Then, according to [23, Theorem 4.3], the operator $R_{\mathrm{loc},*}^{(k)}$ is of strong type (p,p), $1<p<\infty$, and of weak type $(1,1)$ with respect to $x^\delta dx$, for every $\delta\in\mathbb{R}$.

Then we conclude that $R_{\alpha,*}^{(k)}$ defines an operator of strong type (p,p) for $x^\delta dx$ when $1<p<\infty$ and $-\Big(\alpha+\frac{1}{2}+\delta_k\Big)p-1<\delta<\Big(\alpha+\frac{3}{2}\Big)p-1$. We have also that $R_\alpha^{(k)}$ is of weak type $(1,1)$ for $x^\delta dx$ when $-\alpha-\frac{5}{2}\le\delta\le\alpha+\frac{1}{2}$, if k is odd. When k is even the maximal operator $R_{\alpha,*}^{(k)}$ is of

weak type (1,1) with respect to $x^\delta dx$, for $-\alpha-\frac{3}{2}\leq\delta\leq\alpha+\frac{1}{2}$ and $\alpha\neq-\frac{1}{2}$, and for $-1<\delta\leq 0$, when $\alpha=-\frac{1}{2}$.

By using Proposition 3.21 and density arguments we can conclude the proof of this theorem in a standard way. $\square$

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